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Real-valued average consensus over noisy quantized channels

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Abstract— **This paper concerns the average consensus problem with the constraint of quantized communication between nodes. A broad class of algorithms is analyzed, in which the transmission strategy, which decides what value to communicate to the neighbours, can include various kinds of rounding, probabilistic quantization, and bounded noise. The arbitrariness of the transmission strategy is compensated by a feedback mechanism which can be interpreted as a self-inhibitory action. The result is that the average of the nodes state is *not* conserved across iterations, and the nodes do *not* converge to a consensus; however, we show that both errors can be made as small as desired. Bounds on these quantities involve the spectral properties of the graph and can be proved by employing elementary techniques of LTI systems analysis.**

I. INTRODUCTION

Consider the following variation of the average consensus problem [1]: each node in a graph knows a number $x_i(0) \in \mathbb{R}$, and the goal is to drive each node's belief to the initial average α, with the limitation that nodes can communicate only with their neighbours, and that the channel is quantized.

At first sight, this problem looks like a false problem, for if a node can send even only one bit over a channel, then it can send *anything,* by creating an adequate coding. For example, if a resolution of 2^{-8} is needed, then one could consider 8 consecutive bits as a code word, where the i-th bit would represent the i-th bit in the binary expansion of the number being transmitted. With such coding, one can apply the standard consensus algorithms which will achieve a precision of 2^{-8}. If more precision is required, one can use a 16 bit word, and so on. This is true, but the problem can be framed in another way: given a certain quantization, how precise can the consensus be? Previous work always assumed that the achievable precision would have been in the order of the quantization step; instead, we show that consensus can be reached with a precision which is arbitrarily small, at the expense of slower convergence, and without "cheating" by changing the channel coding. For example, it is possible to attain a precision of 2^{-16} even using 8 bit words.

Also note that there are situations in which the channel carries only one bit, and there is no complexity available to change the channel coding. Biological neural networks are such an example. Neurons communicate through trains of spikes, which are a all-or-none, partly probabilistic phenomenon [2], [3]; a spike has a self-inhibitory effect on the spiking neuron and either a excitatory or inhibitory effect on the other connected neurons. Consider the cartoonish representation of a neural network in Fig. 1. Assume some of the neurons are sensible to the same external stimulus (temperature, light intensity, sweetness, etc.) and that we wish to obtain an average of these measures, which we think as real-valued excitation levels. This can be cast as an average consensus problem with $0-1$ communication links, where $\{0,1\}$ can be mapped to $\{\text{no spike}, \text{spike}\}$, subject to "noisy" or "probabilistic" quantization.

More in general, we are interested in the kind of computation that can be implemented by a network of spatially distributed, noisy, slow elements, with limited bandwidth, such as neurons. The evidence from biology suggests that, under these constraints, it is possible to implement fast, robust, and adaptive control systems. Yet we do not have, in our control-systems toolbox, design methods that can work on this computational substrate. In this context, average consensus on a graph is a good toy problem because it clearly has the flavor of "computation", and still it can be solved with tools from linear system theory.

In contrast with previous work (summarized in the next section), we do not consider the quantization strategy as part of the design. Rather, we take the basic discrete-time consensus algorithm [1], and we consider its behavior when the transmitted values are subject to an arbitrary, possibly noisy, quantization strategy. In this paper, we consider any quantization strategy $\boldsymbol{\psi}$, either deterministic or probabilistic, with the condition that $|\boldsymbol{\psi}(x) - x|$ is bounded. We show that the problem can be solved by adding a feedback loop around $\boldsymbol{\psi}$; this is essentially an integrator that compensates for the error in time. When $\boldsymbol{\psi}$ is given the interpretation of a spiking function, the feedback loop can be interpreted as a self-inhibitory action. Using this method, the average of the nodes state is *not* conserved across iterations, and the nodes do *not* converge to a consensus; however, both errors can be made as small as desired, at the cost of slower convergence.

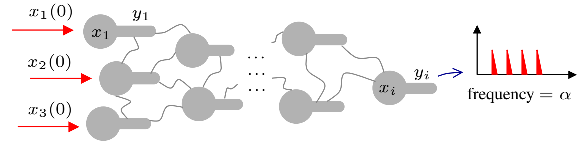

Fig. 1. Cartoonish representation of a biological neural network. A group of neurons is sensible to the same stimulus (for example, light); the activation value plays the same role as the initial node values $x_j(0)$. The problem of averaging the stimulus response can be cast as an average consensus problem with 1-bit channels (spike/no spike). Using the algorithm proposed in this paper, the spiking frequency of the neurons would eventually synchronize approximately to a frequency equal to the average of the external stimuli.

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II. Related work

A consensus algorithm over quantized channels is a two-part strategy:

1) *Communication strategy:* how to decide the value $y_j(k)$ to send to the neighbors.
2) *Update strategy:* how to update one node's value $x_i(k)$ based on the $y_j(k)$ received from its neighbors.

We will be consistent through the paper to use the index j to refer to the communication strategy, and the index i to refer to the update strategy.

Many works dealt with reaching a quantized consensus, in the sense that $x_j(k) \in \mathbb{Z}$, and in the limit the states differ at most by 1; for the analysis of this problem see [4], [5] and references therein.

In [6], [7], [8], [9] the authors consider the problem of reaching a consensus in $\mathbb{R}$ using quantized channels. They propose the following communication strategy:

$$y_j(k) \quad = \quad q(x_j(k))$$

where $q(x)$ rounds x to the nearest integer, and the following is the update strategy:

$$x_i(k+1) = x_i(k) - y_i(k) + \sum_j \mathbf{P}_{i,j}\, y_j(k)$$

where $\mathbf{P}$ is any doubly stochastic matrix with positive diagonal and with the corresponding graph strongly connected.

This algorithm is such that:

- The average of the states is conserved.
- The nodes converge to different values, in general not in $\mathbb{Z}$.
- The disagreement is in the order of the discretization step.

In [10] the authors propose a "probabilistic quantization" scheme. The probabilistic quantization $q^p(x)$ of x is a random variable with the distribution:

$$q^p(x) = \begin{cases} \lceil x \rceil & \text{with probability } x - \lfloor x \rfloor \\ \lfloor x \rfloor & \text{otherwise} \end{cases} \tag{1}$$

They consider the following communication strategy:

$$y_j(k) = q^p(x_j(k))$$

with the following updating strategy:

$$x_i(k+1) = \sum_j \mathbf{P}_{i,j}\, y_j(k)$$

with $\mathbf{P}$ doubly stochastic.

This algorithm is such that:

- The average is not conserved.
- Nodes converge to a consensus $\tau \in \mathbb{Z}$.
- The expected value of τ is α.

III. Proposed strategy

As for the updating strategy, we use the same update rule discussed as the base case in the tutorial paper [1], with the difference is that we use $y_j(k)$, the quantized value transmitted by node j, instead of using the real state $x_j(k)$.

$$x_i(k+1) = x_i(k) + \frac{\eta}{\Delta} \sum_j a_{ij}\,(y_j(k) - x_i(k)) \tag{2}$$

Here a_{ij} is an element of the adjacency matrix and Δ is the maximum degree of the graph. The parameter $\eta \in (0,1)$ influences the convergence speed (and, as we will see, the precision of the consensus).

The communication strategy relies on the definition of an auxiliary state variable $c_j(k) \in \mathbb{R}$, for which initially $c_j(0) = 0$, and a certain function $\boldsymbol{\psi} : \mathbb{R} \to \mathbb{Z}$, or random variable, such that

$$|\boldsymbol{\psi}(y) - y| \leq \beta \tag{3}$$

for some $\beta > 0$.

Our proposed communication strategy is:

$$\begin{cases} y_j(k) &= \boldsymbol{\psi}\left(x_j(k) - c_j(k)\right) \\ c_j(k+1) &= c_j(k) + (y_j(k) - x_j(k)) \end{cases} \tag{4}$$

Note that the auxiliary variable $c_j(k)$ integrates the error in approximating x_j with y_j. This error is then used as a negative feedback for $\boldsymbol{\psi}$. If $\boldsymbol{\psi}$ is interpreted as a spiking

TABLE I
SUMMARY OF CONSIDERED METHODS

Method	Communication strategy	Update strategy	Drift	Disagreement
No quantization	$\boldsymbol{y}(k) = \boldsymbol{x}(k)$	$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) + (\mathbf{P} - \mathbf{I})\boldsymbol{y}(k)$	$d(k) = 0$	$\varphi(k) \to 0$
Carli *et al.*	$\boldsymbol{y}(k) = q(\boldsymbol{x}(k))$	$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) + (\mathbf{P} - \mathbf{I})\boldsymbol{y}(k)$	$d(k) = 0$	$\varphi(k) \to c > 0$
Aysal *et al.*	$\boldsymbol{y}(k) = q^p(\boldsymbol{x}(k))$	$\boldsymbol{x}(k+1) = \mathbf{P}\boldsymbol{y}(k)$	$d(k) \neq 0$	$\varphi(k) \to 0$
Proposed strategy	$\boldsymbol{y}(k) = \boldsymbol{\psi}(\boldsymbol{x}(k) - \boldsymbol{c}(k))$ $\boldsymbol{c}(k+1) = \boldsymbol{c}(k) + (\boldsymbol{y}(k) - \boldsymbol{x}(k))$ for any $\boldsymbol{\psi}$ such that $\lvert\boldsymbol{\psi}(z) - z\rvert \leq \beta$	$\boldsymbol{x}(k+1) = (\mathbf{I} - \frac{\eta}{\Delta}\mathbf{D})\boldsymbol{x}(k) + \frac{\eta}{\Delta}\mathbf{A}\boldsymbol{y}(k)$ for any $\eta \in (0,1)$	$d(k) \leq \eta\beta$	$\varphi(k) \leq c \cdot \eta\beta \frac{\lambda_n\{\mathbf{L}\}}{\lambda_2\{\mathbf{L}\}}$

function, the feedback inhibits the next spike, similarly to the effect of the inhibitory post-synaptic potential in the neuron.

Examples of allowed $\boldsymbol{\psi}$ include:

- *Rounding functions*: Define the function $q : \mathbb{R} \to \mathbb{Z}$ such that $q(x)$ is the integer closest to x. Then one can choose
$$\boldsymbol{\psi}(x) = q(x), \qquad \beta = 0.5$$
- *Ceiling/floor functions*:
$$\boldsymbol{\psi}(x) = \lceil x \rceil, \qquad \beta = 1$$
$$\boldsymbol{\psi}(x) = \lfloor x \rfloor, \qquad \beta = 1$$
- *Threshold-and-fire:* Fire to the next integer if $x - \lfloor x \rfloor$ is over a threshold $t \in (0, 1)$.
$$\boldsymbol{\psi}(x) = \begin{cases} \lceil x \rceil & \text{if } x - \lfloor x \rfloor > t \\ \lfloor x \rfloor & \text{otherwise} \end{cases}, \quad \beta = \max\{t, 1-t\} \tag{5}$$
- *Random rounding:* Choose randomly between the previous and the next integer, according to a fixed probability $p \in (0, 1)$.
$$\boldsymbol{\psi}(x) = \begin{cases} \lceil x \rceil & \text{with probability } p \\ \lfloor x \rfloor & \text{otherwise} \end{cases}, \quad \beta = 1$$
- *Probabilistic quantization*: Use the probabilistic quantization defined as in (1):
$$\boldsymbol{\psi}(x) = \begin{cases} \lceil x \rceil & \text{with probability } x - \lfloor x \rfloor \\ \lfloor x \rfloor & \text{otherwise} \end{cases}, \quad \beta = 1$$

In the case without quantization ($y_j(k) = x_j(k)$), it is possible to obtain, with very mild assumptions, two nice properties of the update strategy (2): the mean is conserved across iterations, and the disagreement tends to zero. This is not true in our case: the mean *is not* conserved, and the states *do not* converge to a consensus. However, we can provide bounds that depend linearly on the parameter η in (2) and therefore can be made as small as desired, and, in particular, much smaller than the quantization.

To measure the performance of the algorithm, we define two error measures. The first is the drift from the mean:
$$d(k) \triangleq \left| \frac{1}{n} \sum_i x_i(k) - \alpha \right|$$
The other is the disagreement among the nodes:
$$\sum_{i,j} a_{ij} \left(x_i(k) - x_j(k)\right)^2 = \boldsymbol{x}(k)^T \mathbf{L} \boldsymbol{x}(k)$$
In this expression $\mathbf{L}$ is the Laplacian of the graph ($\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the degree matrix and $\mathbf{A}$ is the adjacency matrix). Because we are interested in the performance as the number of nodes grows, we look at the average disagreement $\varphi(k)$, which we define as
$$\varphi(k) \triangleq \left[\frac{1}{n\Delta} \sum_{i,j} a_{ij} \left(x_i(k) - x_j(k)\right)^2 \right]^{1/2}$$

TABLE II

SYMBOLS USED IN THIS PAPER

n	number of nodes
$\mathbf{A}$	adjacency matrix
$\mathbf{D}$	degree matrix
d_j	degree of node j
Δ	graph degree
$\mathbf{L}$	Laplacian matrix; $\mathbf{L} = \mathbf{D} - \mathbf{A}$
$\mathbf{P}$	Perron matrix; $\mathbf{P} = \mathbf{I} - \epsilon\mathbf{L}$ with $\epsilon < 1/\Delta$
$x_j(k)$	node state
α	target value: $\alpha = \sum_i x_i(0)/n$
$y_j(k)$	value transmitted by node j to neighbors at time k
$\boldsymbol{\psi}$	generic (noisy) quantization function
$d(k)$	drift
$\varphi(k)$	average disagreement
$q(x)$	nearest integer to x
$q^p(x)$	probabilistic quantization for x
$r(x)$	$\triangleq q(x) - x$
$c_j(k)$	auxiliary state variable
$\mathbf{1}$	$n \times 1$ vector of 1s

Note that we use $n\Delta$ as an approximation to the number of edges; the square root is to obtain a linear measure comparable with $d(k)$.

In the next section we prove the following bounds:
$$d(k) \leq \eta\beta$$
$$\lim_{k\to\infty} \varphi(k) \leq \sqrt{6} \cdot \eta\beta \cdot \frac{\lambda_n\{\mathbf{L}\}}{\lambda_2\{\mathbf{L}\}} \tag{6}$$
In this paper, with abuse of notation, we write $\lim_{k\to\infty} \varphi(k) \leq c$ in the sense that $\varphi(k) \leq c + f(k)$ with $\lim_{k\to\infty} f(k) = 0$; in general, $\varphi(k)$ does not have a limit because it settles on an oscillatory behavior.

Note in (6) the three factors that impact the accuracy of the consensus: $\eta \in (0, 1)$ appears in the update strategy, β depends on the quantization strategy, and $\lambda_n\{\mathbf{L}\}/\lambda_2\{\mathbf{L}\}$ depends on the topology of the graph. The bound on the drift depends only on β and η and is independent of the topology. By choosing a smaller η, we can make these two errors as small as desired; the trade-off is that a smaller η produces slower convergence.

IV. MAIN RESULTS

We briefly recalls the main spectral properties of graphs that we use in the following; for the proofs, see [1] and references therein. See also Table III for a summary of the symbols.

We assume that the graph is undirected and connected. Let $\mathbf{D}$ be the degree matrix and $\mathbf{A}$ the adjacency matrix. Then the Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is symmetric positive semidefinite. The smallest eigenvalue $\lambda_1\{\mathbf{L}\}$ is 0, with eigenvector $\mathbf{1}$: $\mathbf{1}^T\mathbf{L} = \mathbf{0}$. The second smallest eigenvalue $\lambda_2\{\mathbf{L}\}$ is different than zero. The largest eigenvalue $\lambda_n\{\mathbf{L}\}$ is at least Δ and at most 2Δ.

If $\epsilon = \eta/\Delta$, with $\eta \in (0, 1)$ then $\mathbf{P} = \mathbf{I} - \epsilon\mathbf{L}$ is a doubly stochastic matrix. $\mathbf{P}$ and $\mathbf{L}$ have the same eigenvectors. If λ_j is the j-th eigenvalue of $\mathbf{L}$, then the j-th eigenvalue for $\mathbf{P}$ is $\mu_j = 1 - \epsilon\lambda_j$. $\mathbf{1}$ is an eigenvector for the simple eigenvalue 1 of $\mathbf{P}$: $\mathbf{1}^T\mathbf{P} = \mathbf{1}$. Because the other eigenvalues of $\mathbf{P}$ are strictly less than 1, and $\mathbf{1}^T\mathbf{L} = \mathbf{0}$, this implies that $\lim_{k\to\infty} \mathbf{P}^k\mathbf{L} = \mathbf{0}$.

Before stating the main results, we begin with two lemmas. The first describes an invariant quantity of the system.

Lemma 1: Let d_j be the degree of node j and $\epsilon = \eta/\Delta$. Then the following quantity $V(k)$ is invariant:

$$V(k) = \sum_i (x_i(k) - \epsilon d_j c_j(k)) = \mathbf{1}^T\left(\boldsymbol{x}(k) - \epsilon\mathbf{D}\boldsymbol{c}(k)\right)$$

Proof: Notice that $\boldsymbol{y}(k)$ can be written as:

$$\boldsymbol{y}(k) \quad = \quad \boldsymbol{x}(k) + [\boldsymbol{c}(k+1) - \boldsymbol{c}(k)]$$

Hence the dynamics can be rewritten as:

$$\boldsymbol{x}(k+1) = \mathbf{P}\boldsymbol{x}(k) + \epsilon\mathbf{A}(\boldsymbol{c}(k+1) - \boldsymbol{c}(k))$$

Now a straight computation gives:

$$\begin{aligned} V(k+1) &= \mathbf{1}^T(\boldsymbol{x}(k+1) - \epsilon\mathbf{D}\boldsymbol{c}(k+1)) \\ &= \mathbf{1}^T(\mathbf{P}\boldsymbol{x}(k) + \epsilon\mathbf{A}(\boldsymbol{c}(k+1) - \boldsymbol{c}(k)) - \epsilon\mathbf{D}\boldsymbol{c}(k+1)) \\ &= \mathbf{1}^T(\mathbf{P}\boldsymbol{x}(k) - \epsilon\mathbf{D}\boldsymbol{c}(k) + \epsilon\mathbf{L}(\boldsymbol{c}(k+1))) \\ &= \mathbf{1}^T(\boldsymbol{x}(k) - \epsilon\mathbf{D}\boldsymbol{c}(k)) = V(k) \end{aligned}$$

■

Lemma 2: The auxiliary variable $c_j(k)$ used in the feedback loop is bounded: $|c_j(k)| \le \beta$

Proof:

$$\begin{aligned} |c_j(k+1)| &= |c_j(k) + (y_j(k) - x_j(k))| \\ &= |\boldsymbol{\psi}(x_j(k) - c_j(k)) - (x_j(k) - c_j(k))| \\ &= |\boldsymbol{\psi}(z) - z| \le \beta \end{aligned}$$

■

Given the previous lemma, the bound on the drift is an easy consequence.

Proposition 1: The drift is bounded:

$$d(k) \le \eta\beta \tag{7}$$

Proof: Notice that

$$\alpha = \frac{1}{n}V(0) = \frac{1}{n}V(k) = \frac{1}{n}\mathbf{1}^T\left(\boldsymbol{x}(k) - \epsilon\mathbf{D}\boldsymbol{c}(k)\right),$$

and thus

$$\begin{aligned} \left|\frac{1}{n}\mathbf{1}^T\boldsymbol{x}(k) - \alpha\right| &= \left|\epsilon\frac{1}{n}\mathbf{1}^T\mathbf{D}\boldsymbol{c}(k)\right| \le \epsilon\frac{1}{n}\mathrm{Tr}(\mathbf{D})\beta \\ &\le \epsilon\frac{1}{n}(n\Delta)\beta = \epsilon\Delta\beta = \eta\beta \end{aligned}$$

■

Proposition 2: Eventually, the disagreement is bounded by ϵ:

$$\lim_{k\to\infty} \varphi(k) \le \sqrt{6}\cdot\eta\beta\frac{\lambda_n\{\mathbf{L}\}}{\lambda_2\{\mathbf{L}\}} \tag{8}$$

Proof: The dynamics of the system can be written as

$$\boldsymbol{x}(k+1) = \mathbf{P}\boldsymbol{x}(k) + \epsilon\mathbf{A}\boldsymbol{c}(k+1) - \epsilon\mathbf{A}\boldsymbol{c}(k),$$

where $\epsilon = \eta/\Delta$. In this proof, we consider $\mathbf{A}\boldsymbol{c}(k)$ as a disturbance input to the system $\boldsymbol{x}(k+1) = \mathbf{P}\boldsymbol{x}(k)$. Because $\boldsymbol{c}(k)$ is bounded (Lemma 2) and $\mathbf{P}$, being doubly stochastic, has a "mixing" effect, the disturbance is filtered by the dynamics, and the final effect on the consensus can be bounded.

Note that $\epsilon\mathbf{A}\boldsymbol{c}(k)$ is added to the state at time $\boldsymbol{x}(k)$ (with a plus sign) and to $\boldsymbol{x}(k+1)$ (with a minus sign). The contribution to $\boldsymbol{x}(k+1)$ is globally $+\mathbf{P}\epsilon\mathbf{A}\boldsymbol{x}(k) - \epsilon\mathbf{A}\boldsymbol{c}(k) = \epsilon(\mathbf{P} - \mathbf{I})\mathbf{A}\boldsymbol{c}(k) = -\epsilon^2\mathbf{L}\mathbf{A}\boldsymbol{c}(k)$. Consequently, the state admits the following closed form expression:

$$\boldsymbol{x}(k) = \mathbf{P}^k\boldsymbol{x}(0) + \epsilon\mathbf{A}\boldsymbol{c}(k) - \epsilon^2\sum_{\tau=1}^{k-1}\mathbf{P}^{k-\tau}\mathbf{L}\mathbf{A}\boldsymbol{c}(\tau) - \epsilon\mathbf{P}^k\mathbf{A}\boldsymbol{c}(0)$$

We want to compute the limit of the disagreement function $\boldsymbol{x}(k)^T\mathbf{L}\boldsymbol{x}(k)$ as $k \to \infty$. Note that it is composed by 6 terms:

$$\begin{aligned} &\boldsymbol{x}(k)^T\mathbf{L}\boldsymbol{x}(k) = && (9) \\ &\quad \boldsymbol{x}(0)^T\mathbf{P}^k\mathbf{L}\mathbf{P}^k\boldsymbol{x}(0) + && (10) \\ &\quad \epsilon^2\boldsymbol{c}(k)^T\mathbf{A}\mathbf{L}\mathbf{A}\boldsymbol{c}(k) + && (11) \\ &\quad \epsilon^4\sum_{m=1}^{k-1}\sum_{\tau=1}^{k-1}\boldsymbol{c}(\tau)^T\mathbf{A}\mathbf{L}\mathbf{P}^{k-m}\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\mathbf{A}\boldsymbol{c}(\tau) + && (12) \\ &\quad \epsilon\boldsymbol{x}(0)^T\mathbf{P}^k\mathbf{L}\mathbf{A}\boldsymbol{c}(k) + && (13) \\ &\quad -\epsilon^2\boldsymbol{x}(0)^T\mathbf{P}^k\mathbf{L}\sum_{\tau=1}^{k-1}\mathbf{P}^{k-\tau}\mathbf{L}\mathbf{A}\boldsymbol{c}(\tau) + && (14) \\ &\quad -\epsilon^3\boldsymbol{c}(k)^T\mathbf{A}\sum_{\tau=1}^{k-1}\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\mathbf{A}\boldsymbol{c}(\tau) && (15) \end{aligned}$$

Because $\mathbf{P}^k\mathbf{L} \to \mathbf{0}$, the terms (10), (13), (14) disappear. The term (11) can be bounded as follows:

$$\begin{aligned} |\epsilon^2\boldsymbol{c}(k)^T\mathbf{A}\mathbf{L}\mathbf{A}\boldsymbol{c}(k)| &\le \beta^2\epsilon^2\max_{||\boldsymbol{u}||_\infty\le 1}|\boldsymbol{u}^T\mathbf{A}\mathbf{L}\mathbf{A}\boldsymbol{u}| \\ &\le \beta^2\epsilon^2\Delta^2\max_{||\boldsymbol{u}||_\infty\le 1}|\boldsymbol{u}^T\mathbf{L}\boldsymbol{u}| \\ &\le \beta^2\epsilon^2\Delta^2 n\lambda_n\{\mathbf{L}\} \end{aligned}$$

The term (15) can be bounded as

$$\begin{aligned} &|\epsilon^3\boldsymbol{c}(k)^T\mathbf{A}\sum_{\tau=1}^{k-1}\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\mathbf{A}\boldsymbol{c}(\tau)| \quad \le \\ &\epsilon^3\sum_{\tau=1}^{k-1}|\boldsymbol{c}(k)^T\mathbf{A}\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\mathbf{A}\boldsymbol{c}(\tau)| \quad \le \\ &\beta^2\epsilon^2\Delta^2 n\sum_{\tau=1}^{k-1}\lambda_{\max}\left\{\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\right\} \end{aligned}$$

Recall that $\mathbf{P}$ and $\mathbf{L}$ have the same eigenvectors; hence the typical eigenvalue for the matrix $\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}$ has the value

$$\lambda_i\left\{\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\right\} = \lambda_i\{\mathbf{L}\}^2\left(1 - \epsilon\lambda_i\{\mathbf{L}\}\right)^{k-\tau}$$

This expression can be bounded by choosing the largest eigenvalue λ_n for the first factor, and the smallest non-zero eigenvalue λ_2 for the second factor.

$$\lambda_{\max}\left\{\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\right\} \le \lambda_n\{\mathbf{L}\}^2\left(1 - \epsilon\lambda_2\{\mathbf{L}\}\right)^{k-\tau}$$

The sum of the series can be computed as:

$$\sum_{\tau=1}^{k-1}(1-r)^{k-\tau} \quad = \quad \frac{(1-r)}{r}\left[1-(1-r)^{k-1}\right]$$

Hence for the sixth term:

$$\left|\epsilon^3 \boldsymbol{c}(k)^T \mathbf{A}\sum_{\tau=1}^{k-1}\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\mathbf{A}\boldsymbol{c}(\tau)\right| \leq$$

$$\beta^2\epsilon^2\Delta^2 n\frac{\lambda_n^2\{\mathbf{L}\}}{\lambda_2\{\mathbf{L}\}}\left(1-\epsilon\lambda_2\{\mathbf{L}\}\right)\left[1-(1-\epsilon\lambda_2\{\mathbf{L}\})^{k-1}\right]$$

For the same reasons, but with longer computations, the remaining term can be bounded as follows:

$$\left|\epsilon^4\sum_{m=1}^{k-1}\sum_{\tau=1}^{k-1}\boldsymbol{c}(\tau)^T\mathbf{A}\mathbf{L}\mathbf{P}^{k-m}\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\mathbf{A}\boldsymbol{c}(\tau)\right| \quad \leq$$

$$\beta^2\epsilon^4 n\Delta^2\sum_{m=1}^{k-1}\sum_{\tau=1}^{k-1}\lambda_{\max}\left\{\mathbf{L}\mathbf{P}^{k-m}\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\right\}$$

We find again that

$$\lambda_{\max}\left\{\mathbf{L}\mathbf{P}^{k-m}\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\right\} \quad \leq$$

$$\lambda_n^3\left\{\mathbf{L}\right\}\left(1-\epsilon\lambda_2\left\{\mathbf{L}\right\}\right)^{(k-m)+(k-\tau)}$$

and for the series:

$$\sum_{m=1}^{k-1}\sum_{\tau=1}^{k-1}\left(1-\epsilon\lambda_2\left\{\mathbf{L}\right\}\right)^{(k-m)+(k-\tau)} \quad =$$

$$\frac{(1-\epsilon\lambda_2\{\mathbf{L}\})^2}{\epsilon^2\lambda_2\{\mathbf{L}\}^2}\left[1-(1-\epsilon\lambda_2\{\mathbf{L}\})^{k-1}\right]^2$$

And hence the bound for the sixth term is

$$\left|\epsilon^4\sum_{m=1}^{k-1}\sum_{\tau=1}^{k-1}\boldsymbol{c}(\tau)^T\mathbf{A}\mathbf{L}\mathbf{P}^{k-m}\mathbf{L}\mathbf{P}^{k-\tau}\mathbf{L}\mathbf{A}\boldsymbol{c}(\tau)\right| \leq$$

$$\beta^2\epsilon^2 n\Delta^2\frac{\lambda_n^3\{\mathbf{L}\}}{\lambda_2^2\{\mathbf{L}\}}\left(1-\epsilon\lambda_2\{\mathbf{L}\}\right)^2\left[1-(1-\epsilon\lambda_2\{\mathbf{L}\})^{k-1}\right]^2$$

Finally, the limit for the disagreement function is

$$\lim_{k\to\infty}\boldsymbol{x}(k)^T\mathbf{L}\boldsymbol{x}(k) \leq \beta^2\epsilon^2 n\Delta^2\lambda_n\{\mathbf{L}\}\left(1+\frac{\lambda_n\{\mathbf{L}\}}{\lambda_2\{\mathbf{L}\}}+\frac{\lambda_n^2\{\mathbf{L}\}}{\lambda_2^2\{\mathbf{L}\}}\right)$$

Note that $\lambda_n/\lambda_2 \geq 1$; a good approximation is

$$\left(1+\frac{\lambda_n\{\mathbf{L}\}}{\lambda_2\{\mathbf{L}\}}+\frac{\lambda_n^2\{\mathbf{L}\}}{\lambda_2^2\{\mathbf{L}\}}\right) \leq 3\frac{\lambda_n^2\{\mathbf{L}\}}{\lambda_2^2\{\mathbf{L}\}}$$

At this point, use the fact that $\lambda_n\{\mathbf{L}\} \leq 2\Delta$:

$$\lim_{k\to\infty}\boldsymbol{x}(k)^T\mathbf{L}\boldsymbol{x}(k) \leq 6\beta^2\epsilon^2 n\Delta^3\frac{\lambda_n^2\{\mathbf{L}\}}{\lambda_2^2\{\mathbf{L}\}}$$

Hence for the average disagreement:

$$\lim_{k\to\infty}\varphi(k) \leq \sqrt{6}\cdot\eta\beta\frac{\lambda_n\{\mathbf{L}\}}{\lambda_2\{\mathbf{L}\}}$$

■

V. SIMULATIONS

Fig. 2 shows an example run of our algorithm, with $\boldsymbol{\psi} = q$, on a circular graph composed of $n = 10$ nodes, for $\eta = 0.01$ and $\eta = 0.05$. The states eventually seem to converge to a periodic function (but note that we did not prove such result). These results are quantitatively similar for other choices of deterministic $\boldsymbol{\psi}$; while, of course, probabilistic quantizations do not settle on a periodic behavior.

Table VI shows the result of comparing our algorithm to the one proposed by Carli *et al.* on a set of canonical graphs (star-shaped, ring, path, complete), for two sizes of the graphs ($n = 10, 30$). We set the initial value of node i to be $x_i(0) = \pi i$; the asymptotic results are qualitatively unchanged if one chooses random initial values. The simulation is run for 10,000 time steps, which is enough for the methods to reach the stationary behavior. Because an equilibrium is not reached, we report in the table the worst values of the drift and the disagreement as recorded in the last 100 steps. The results of our algorithm appear substantially better than the algorithm of Carli *et al.* However, we observe that the bounds we found are very loose, especially the bound on the disagreement given by Proposition 2.

VI. CONCLUSIONS AND FUTURE WORK

This paper showed how real-valued consensus can be reached (up to an arbitrary accuracy) even in the case that the channels are quantized, and the quantization function is noisy. The method consists in wrapping a negative feedback loop around the quantization function; this has a loose similarity to the self-inhibitory action of spiking neurons.

The algorithm presented in this paper seems to be effective, but the bounds derived, while they show qualitatively that the errors can be made as small as desired by varying the parameter η, are not quantitatively satisfactory, because they greatly overestimate the errors. The reason is that in the derivation of Proposition 2 we treated the quantization function essentially as an arbitrary bounded disturbance. The analysis is particularly pessimistic for the ring and path graphs, because $\lambda_n/\lambda_2 \sim n^2$, while, in practice, we observed that the asymptotic disagreement seems to be largely independent of the number of nodes. These results can be improved by either focusing on one particular quantization function instead of the broad class we considered, or on one particular class of graphs. The speed of convergence must be investigated further. Far from convergence, before the quantization error becomes relevant, the convergence appears to be exponential, and this should be easily proved. When the quantization error is dominant, the analysis complicates because of the nonlinearity and stochasticity of $\boldsymbol{\psi}$; in the case of deterministic quantizations, the system seems to tend to a periodic orbit, but we do not have a proof of this yet.

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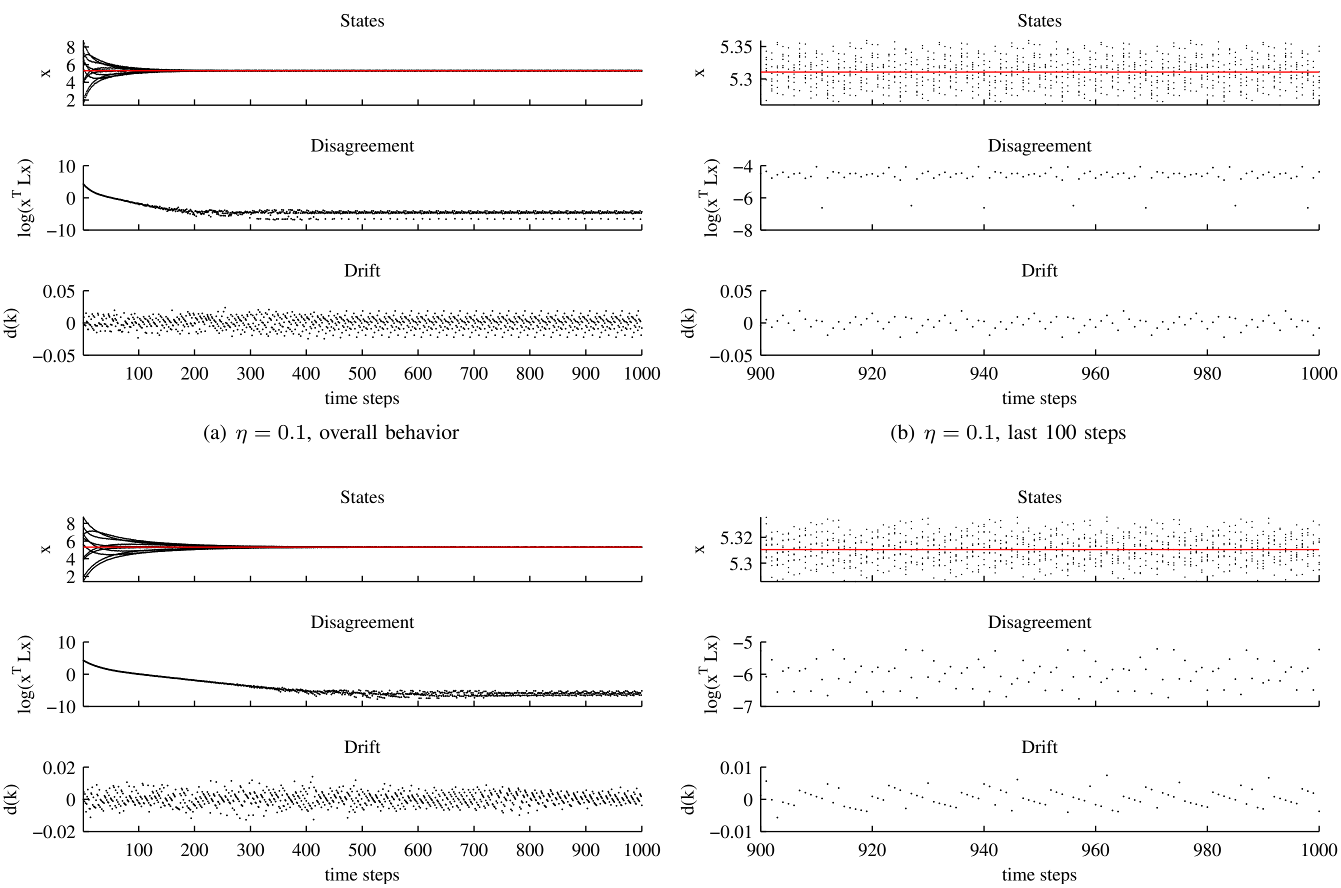
TABLE III

SIMULATIONS

graph	nodes	Δ	$\lambda_n \mathbf{L}$	$\lambda_2 \mathbf{L}$	λ_n/λ_2	Carli et al.	Proposed			
						$\varphi(k)$	$d(k)$	bound	$\varphi(k)$	bound
star	10	$n-1$	n	1	n	0.01263	0.00493	0.050	0.00003262	1.22474
	30					0.00341	0.00166	0.050	0.00000153	3.67423
complete	10	$n-1$	n	n	1	0.05904	0.00756	0.050	0.00001391	0.12247
	30					0.03939	0.00320	0.050	0.00000117	0.12247
ring	10	2	4	$2-2\cos\left(\frac{2\pi}{n}\right)$	$\sim n^2$	0.04214	0.01512	0.050	0.00113779	1.28256
	30					0.02467	0.00740	0.050	0.00076440	11.20924
path	10	2	$2+2\cos\left(\frac{\pi}{n}\right)$	$2-2\cos\left(\frac{\pi}{n}\right)$	$\sim n^2$	0.03594	0.01767	0.050	0.00061120	4.88225
	30					0.00911	0.01767	0.050	0.00136018	44.59170

Fig. 2. Consensus on a ring graph with $n = 10$ nodes.



(a) $\eta = 0.1$, overall behavior

(b) $\eta = 0.1$, last 100 steps

(c) $\eta = 0.05$, overall behavior

(d) $\eta = 0.05$, last 100 steps

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